DSP Lec 6

* Inverse Discrete Fourier Transform (IDFT)

Note

Fourier Transform (Cont. Form)

$$x(t) \longrightarrow X(jw) = \int_{t=-\infty}^{\infty} x(t)\, e^{-jwt}$$

$$X(jw) \xrightarrow[\int^{-1}]{IFT} x(t) = \frac{1}{2\pi} \int_{w=-\infty}^{\infty} X(jw)\, e^{jwt}\, dw$$

$\int^{-1}$ → inverse FT

$$x(n) \xrightarrow{DFT} X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi K}{N}n}$$

$x(n)$ → Discrete time sequence

$X(K)$ → Discrete Fourier Transform

$X(K) \xrightarrow{IDFT} x(n)$ (Discrete time sequence)

$$X(K) \xrightarrow[N]{IDFT} x(n)$$

$$X(K) \xrightarrow[N]{\text{IDFT}} \boxed{x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{J\frac{2\pi K n}{N}}}$$

* if the discrete time sequence $x(n)$ is periodic every N samples, then the discrete Fourier Transform is also periodic.

* If $X(K)$ is periodic, the discrete time sequence $x(n)$ is also periodic.

$$\underbrace{x(n)}_{\text{Required}} = \frac{1}{N}\sum_{K=0}^{N-1} \underbrace{X(K)\, e^{J\frac{2\pi K n}{N}}}_{\text{given DFT } X(K)}$$

Required → discrete time sequence $x(n)$

$$W_N = e^{-J\frac{2\pi}{N}} \Rightarrow \text{twiddle factor}$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$$

$$x_N = \frac{1}{N} * \left[W_N^* \right] X_N$$

$$x_N = \begin{bmatrix} x(n=0) \\ x(n=1) \\ \vdots \\ x(N-1) \end{bmatrix} ; \quad X_N = \begin{bmatrix} X(K=0) \\ X(K=1) \\ \vdots \\ X(K=N-1) \end{bmatrix}$$

$x_N \rightarrow$ discrete time sequence in vector form

$X_N \rightarrow$ DFT in vector form

$$\left[W_N^* \right] = \left[W_N^{-Kn} \right]_{N*N}$$

<u>assume</u> $N = 3$

$$\left[W_3^* \right] = \begin{array}{c} \\ K=0 \\ K=1 \\ K=2 \end{array} \begin{array}{c} \begin{array}{ccc} n=0 & n=1 & n=2 \end{array} \\ \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^{-1} & W_3^{-2} \\ W_3^0 & W_3^{-2} & W_3^{-4} \end{bmatrix}_{N*N} \end{array}$$

$$e^{J\theta} = \cos(\theta) + J\sin(\theta)$$

$$e^{-J\theta} = \cos(\theta) - J\sin(\theta)$$

$$[W_N^*] = [W_N]$$

مع عكس إشارة الجزء التخيلي

$$[W_3^*] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5 + J\frac{\sqrt{3}}{2} & -0.5 - J\frac{\sqrt{3}}{2} \\ 1 & -0.5 - J\frac{\sqrt{3}}{2} & -0.5 + J\frac{\sqrt{3}}{2} \end{bmatrix}$$

For $N = 4$

$$[W_4^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & J \end{bmatrix}$$

$\boxed{Ex}$ find IDFT of $X(K) = \underbrace{\{6, -2+J2, -2, -2-J2\}}_{N=4}$

$$x_4 = \frac{1}{4}\left[W_4^{*}\right] X_4$$

$$x_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & J \end{bmatrix}\begin{bmatrix} 6 \\ -2+J2 \\ -2 \\ -2-J2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6-2+J2-2-2-J2 \\ 6-2J-2+2+J2-2 \\ 6+2-J2-2+2+J2 \\ 6+2J+2+2-2J+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix}$$

$$x_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of DFT

[1] Periodic Property

$x(n)$ is periodic then $X(K)$ is also Periodic

$$X(K) = X(K+N)$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-J\frac{2\pi K n}{N}}$$

Put $K \longrightarrow K+N$

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, e^{-J\frac{2\pi (K+N) n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-J\frac{2\pi K n}{N}} \cdot e^{-J\frac{2\pi N n}{N}}$$

$$e^{-J 2\pi n} = \cos(2\pi n) - J \sin(2\pi n)$$

$$n = 0, 1, \ldots \quad N-1$$

$$\therefore e^{-J 2\pi n} = 1$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, e^{-J\frac{2\pi K n}{N}}$$

$$= X(K)$$

2 Linear Property

$$x_1(n) \xrightarrow[N]{DFT} X_1(K)$$

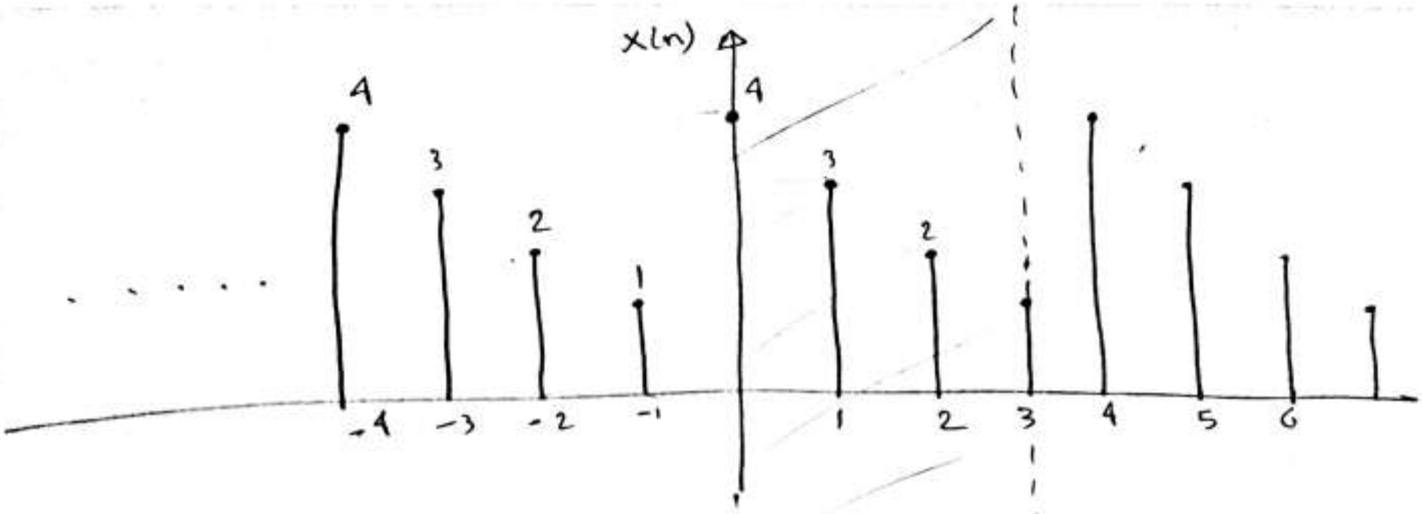
$$x_2(n) \xrightarrow[N]{DFT} X_2(K)$$

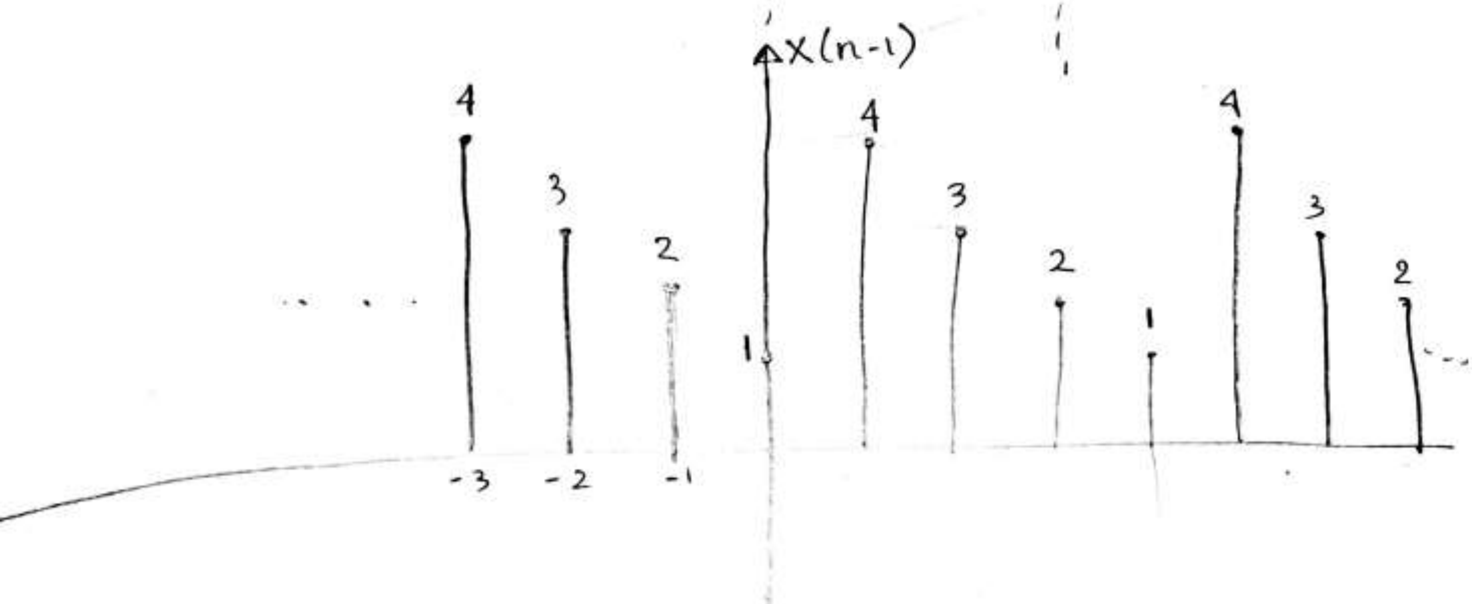
$$a_1 x_1(n) + a_2 x_2(n) \xrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$$

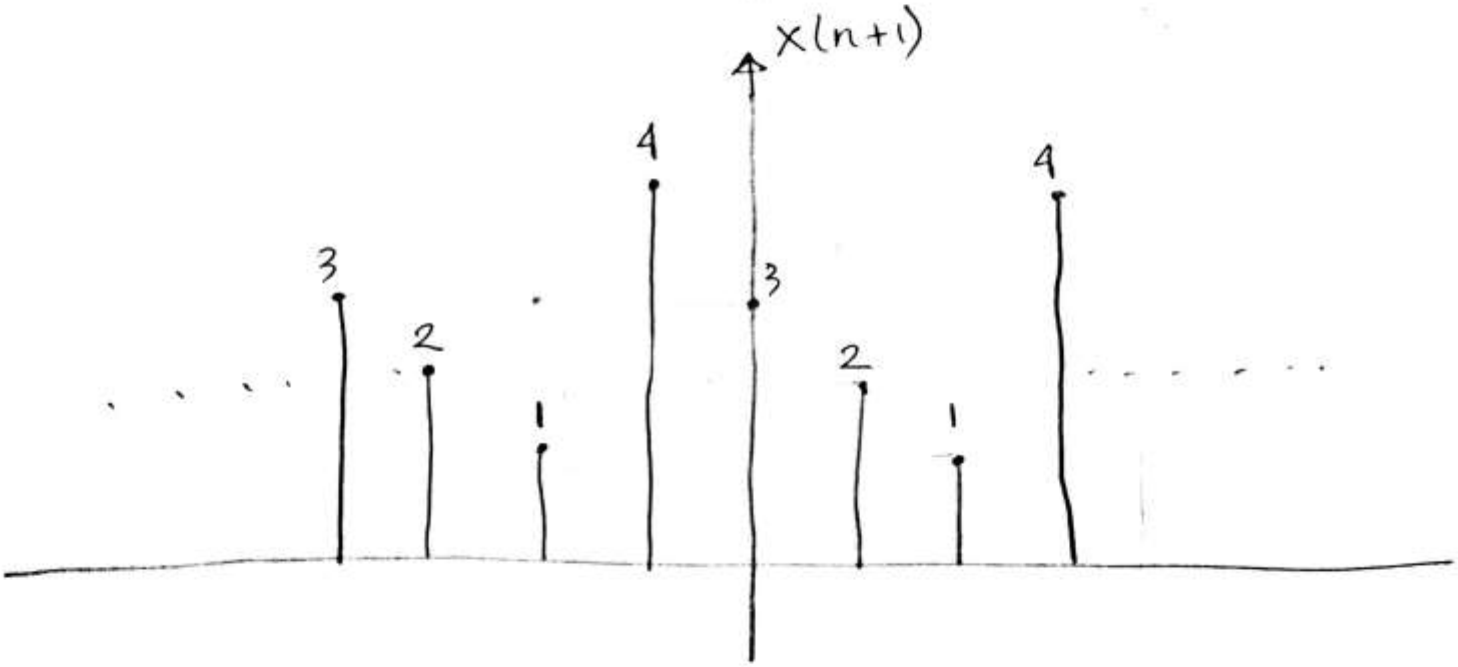
3 Circular shift

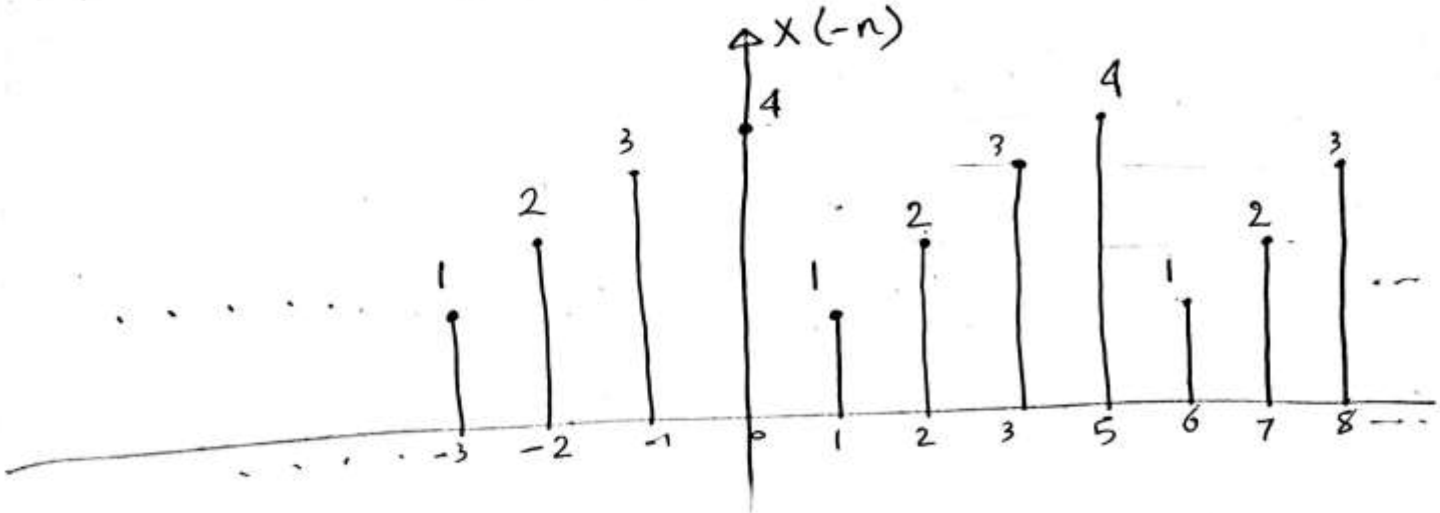
* If the sequence is Periodic, the shift is Called Circular shift

EX $x(n) = \{4, 3, 2, 1\}$ is Periodic sequence.


x(n)
4
3
2
1
4
3
2
-4
-3
-2
-1
1
2
3
4
5
6


x(n-1)
4
3
2
1
4
3
2
1
4
3
2
-3
-2
-1


x(n+1)
3
2
1
4
3
2
1
4



For $\boxed{0 \le n \le N-1}$ one Period

ex) $x(n) = \{4,3,2,1\} = \{x(0), x(1), x(2), x(N-1=3)\}$

• find $x((n-1))_4$ → circular shift (عدد التكرار بتاع ال shift)

$x((n-1))_4 = \{1,4,3,2\} = \{x(3), x(0), x(1), x(2)\}$

shift Rotate to right

$x((n+1))_4 = \{3,2,1,4\}$

shift rotate to left

$x((-n))_4 = \{4,1,2,3\}$

→ in folding ← تثبت أول قيمة وتعكس الباقي

$\boxed{Ex}$ $x(n) = \{1, 0.5, 0, 1, 2\}$

find ① $x((-n))_5 \Longrightarrow \{1, 2, 1, 0, 0.5\}$

$\rightarrow x((n-1))_5 = \{2, 1, 0.5, 0, 1\}$

② $x((n-2))_5$ $\quad x((n-1))_5 = \{1, 2, 1, 0.5, 0\}$

③ $x((1-n))_5 \longrightarrow = \{$

④ $x((n+2))_5 \longrightarrow = \{0, 1, 2, 1, 0.5\}$

⑤ $x((2-n))_5$

③ $x((1-n))_5 = x((-(n-1)))_5$

$x((-n))_5 = \{1, 2, 1, 0, 0.5\}$

$x((1-n))_5 = \{0.5, 1, 2, 1, 0\}$

④ $x((2-n))_5 = \{0, 0.5, 1, 2, 1\}$